Geometric Quantization from a Coherent State Viewpoint

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Abstract

A fully geometric procedure of quantization that utilizes a natural and necessary metric on phase space is reviewed and briefly related to the goals of the program of geometric quantization.

Introduction and Background

Purpose and achievements

The goal of the present work is to present a *conceptually simple, geometric prescription for quantization*. Such a goal has been, and continues to be, the subject of a number of research efforts. In the present paper we shall look at the problems faced by this program from the point of view of coherent states. It will be our conclusion that the symplectic geometry of classical mechanics, augmented by a natural, and even necessary, metric on the classical phase space, are the only essential ingredients to provide a quantization scheme that is fully geometric in character and one that can be expressed in a coordinate-free form. In a surprising sense we shall see that coherent states—far from being optional—are in fact an automatic consequence of this process of quantization. It is fair to say at the outset that we will have very

little to say about the *methods* used in the program of geometric quantization; instead, what we adopt is the *goal* of this program. As will be clear our methods, especially for dynamical questions, are quite different. Moreover, we shall succeed in correctly quantizing a large number (a dense set) of Hamiltonians. In that sense the procedures to be described offer a fully satisfactory solution to the program of geometric quantization, namely to provide an intrinsic and coordinate-free prescription for quantization which agrees with known quantum mechanical results.

Let us start our story with a brief overview of just what it is about classical and quantum mechanics that gives rise in the first place to the problem that the program of geometric quantization addresses. For ease of exposition we shall assume that our phase space admits global coordinates unless explicitly stated otherwise. And for pedagogical reasons we shall focus on just a single degree of freedom; if we don't understand one degree of freedom well, then we probably don't understand much else!

Symplectic manifolds and classical mechanics

For a single degree of freedom, phase space is a two-dimensional manifold M equipped with a *symplectic form* ω, which is a closed, nondegenerate two form. As a closed form it is locally exact, that is $\omega = d\theta$, where θ is a one form and d denotes the operation of exterior derivative. This one form is not unique and it follows from the fact that $d^2 = 0$ that we can replace θ by $\theta + dG$, where G is a zero form or scalar function. For a general G it follows that

$$d(\theta + dG) = d\theta + d^2G = d\theta = \omega \ .$$

In addition we introduce a scalar function H on the manifold M which takes on a unique value at each point $x \in M$. Now consider a path $x(t)$, $0 \equiv t' \leq t \leq t'' \equiv T$, in the manifold M parameterized by the variable t ("time"), so that θ, G, and H all become functions of t through their dependence on the phase-space points along the path. Thus we can construct an action integral according to

$$I = \int (\theta + dG - H\,dt) = \int (\theta - H\,dt) + G'' - G' \ ,$$

and extremal variations, which hold both end points $x(t')$ and $x(t'')$ fixed, lead to the sought for classical trajectory through the manifold. The associated

equations of motion do not involve G. In that sense there is an *equivalence class* of actions all of which lead to the same classical equations of motion and thus the same classical trajectory.

This scenario may be given a more familiar look with the introduction of canonical coordinates. The Darboux theorem assures us that charts of local canonical coordinates p and q exist for which $\theta = p\,dq$, $\omega = dp \wedge dq$, $G = G(p,q)$, and $H = H(p,q)$. In these terms

$$I = \int [p\,dq + dG(p,q) - H(p,q)\,dt] = \int [p\,dq - H(p,q)\,dt] + G(p'',q'') - -G(p',q')\,,$$

where $p', q' \equiv p(0), q(0)$, etc. Insisting that the vanishing of the first-order variation holding the end points p'', q'' and p', q' fixed characterizes the classical trajectory and leads to the usual Hamiltonian equations of motion, namely

$$\dot{q} = \frac{\partial H(p,q)}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H(p,q)}{\partial q}\,.$$

These equations of motion are independent of G, and thus we have an equivalence class of actions all of which lead to the same classical equations of motion.

New canonical coordinates, say $\overline{p}$ and $\overline{q}$, are invariably related to the original canonical coordinates through the one form

$$\overline{p}\,d\overline{q} = p\,dq + dF(\overline{q}, q)$$

for some function F. It follows, therefore, that for a general G and F, one may always find a function $\overline{G}$ such that

$$I = \textstyle\int [\overline{p}\,d\overline{q} + d\overline{G}(\overline{p},\overline{q}) - \overline{H}(\overline{p},\overline{q})\,dt] = \int [\overline{p}\,d\overline{q} - \overline{H}(\overline{p},\overline{q})\,dt] + \overline{G}(\overline{p}'',\overline{q}'') - \overline{G}(\overline{p}',\overline{q}')\,,$$

where $\overline{H}(\overline{p},\overline{q}) = H(p,q)$. Extremal variation of this version of the action holding the end points fixed leads to Hamilton's equations expressed in the form

$$\dot{\overline{q}} = \frac{\partial \overline{H}(\overline{p},\overline{q})}{\partial \overline{p}}\,, \qquad \dot{\overline{p}} = -\frac{\partial \overline{H}(\overline{p},\overline{q})}{\partial \overline{q}}\,.$$

This form invariance of Hamilton's equations among canonical coordinates is what distinguishes this family of coordinate systems in the first place. Such form invariance is the clue that after all there is something of a geometrical

nature underlying this structure, and that geometry is in fact the symplectic geometry briefly discussed above.

It is noteworthy that a given Hamiltonian expressed, say, by $H(p,q) = \frac{1}{2}(p^2+q^2)+q^4$, in one set of canonical coordinates, could, in a new set of canonical coordinates, be expressed simply as $\overline{H}(\overline{p},\overline{q}) = \overline{p}$. If we interpret the first expression as corresponding physically to an anharmonic oscillator, a natural question that arises is how is one to "read" out of the second expression that one is dealing with an anharmonic oscillator. This important question will figure significantly in our study!

Old quantum theory

In the old quantum theory one approximately quantized energy levels by the Bohr-Sommerfeld quantization scheme in which

$$\oint p\,dq = (n+\tfrac{1}{2})2\pi\hbar\ ,$$

where the integral corresponds to a closed contour in phase space at a constant energy value. This integral is to be taken in some canonical coordinate system, but which one? Since two canonical coordinate systems are related by

$$\overline{p}\,d\overline{q} = p\,dq + dF(\overline{q},q)\ ,$$

it follows when M is simply connected that

$$\oint \overline{p}\,d\overline{q} = \oint p\,dq\ ,$$

and so which canonical coordinates are used doesn't matter—they all give the same result. In coordinate-free language this remark holds simply because

$$\oint p\,dq = \int dp\wedge dq = \int d\overline{p}\wedge d\overline{q} \equiv \int \omega\ .$$

Thus we also learn one answer to the question posed above, namely the physics of the mathematical expression for the Hamiltonian given simply by $\overline{p}$ is coded into the orbits and into the coordinate-invariant phase-space areas captured by $\int\omega$ for each value of the energy.

New quantum theory

The royal route to quantization, according to Schrödinger for example, consists first of introducing a Hilbert space of functions $\psi(x)$, defined for $x \in \mathbb{R}$, each of which satisfies $\int |\psi(x)|^2 \, dx < \infty$. Next the classical phase-space variables p and q are "promoted" to operators, $p \rightarrow -i\hbar\partial/\partial x$ and $q \rightarrow x$, which act by differentiation and multiplication, respectively. More general quantities, such as the Hamiltonian, become operators according to the rule

$$H(p, q) \rightarrow \mathcal{H} = H(-i\hbar\partial/\partial x, x) \; ,$$

an expression that may have ordering ambiguities but which we will ignore in favor of the deeper question: In which canonical coordinate systems does such a quantization procedure work? Dirac's answer is: "This assumption is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a Cartesian system of axes and not to more general curvilinear coordinates." [1]. In other words, the correctness of the Schrödinger rule of quantization depends on using the right coordinates, namely Cartesian coordinates. It is worth emphasizing that Cartesian coordinates can only exist on a *flat space*. Likewise in the prescription of Heisenberg which asks (among other things) that the classical canonical variables p and q be replaced by operators P and Q that satisfy the commutation relation $[Q, P] = i\hbar$; this rule must also be applied only in Cartesian coordinates.

An analogous feature is evident in the Feynman phase-space path integral formally given by

$$\mathcal{M} \int e^{(i/\hbar)\int_0^T [p\dot{q} - H(p,q)] \, dt} \, \mathcal{D}p \, \mathcal{D}q \; .$$

Despite the appearance of this expression as being covariant under a change of canonical coordinates, it is, as commonly known, effectively undefined. One common way to define this expression is by means of a lattice formulation one form of which is given by

$$\lim_{N\rightarrow\infty} M \int \exp\{\tfrac{i}{\hbar}\Sigma_0^N [p_{l+\frac{1}{2}}(q_{l+1} - q_l) - \epsilon H(p_{l+\frac{1}{2}}, \tfrac{1}{2}(q_{l+1} + q_l))]\} \, \Pi_0^N \, dp_{l+\frac{1}{2}} \, \Pi_1^N \, dq_l$$

where $\epsilon \equiv T/(N+1)$, $q_{N+1} \equiv q''$, $q_0 \equiv q'$, and $M = (2\pi\hbar)^{-(N+1)}$ is a suitable normalization factor. The indicated limit is known to exist for a large class of Hamiltonians leading to perfectly acceptable (Weyl-ordered) quantizations. However, it is clear that unlike the formal continuum path

integral, this lattice formulation is *not* covariant under canonical coordinate transformations; in other words, this lattice expression will lead to the correct quantum mechanics only in a limited set of canonical coordinates, namely the Cartesian set mentioned by Dirac.

This then is the dilemma that confronts us. The "new" quantization of Schrödinger, Heisenberg, and Feynman—the *correct* quantization from all experimental evidence—seems to depend on the choice of coordinates. This is clearly an unsettling state of affairs since nothing physical, like quantization, should depend on something so arbitrary as the choice of coordinates!

Geometric quantization

There are two attitudes that may be taken toward this apparent dependence of the very act of quantization on the choice of coordinates. The first view would be to acknowledge the "Cartesian character" that is seemingly part of the procedure. The second view would be to regard it as provisional and seeks to find a quantization formulation that eliminates this apparently unphysical feature of the current approaches. There is much to be said for this second view. After all Newton's equations for particle dynamics expressed originally in Cartesian coordinates may be given a tensorial formulation that is valid in all coordinate systems. There seems to be no apparent reason that some similar reformulation of the usual quantization procedures may not do the same for quantum mechanics.

The goal of eliminating the dependence on Cartesian coordinates in the standard approaches is no doubt one of the motivations for several programs such as geometric quantization [2], deformation quantization [3], etc. In the first of these programs, for example, one finds the basic ingredients: (i) prequantization and (ii) polarization (real and complex), which define the framework, i.e, the *kinematics*, and (iii) one of several proposals to deal with *dynamics*. Despite noble efforts, it is not unfair to say that to date only a very limited class of dynamical systems can be treated in the geometric quantization program which also conform with the results of quantum mechanics.

The approach that we shall adopt takes the other point of view seriously, namely that the "Cartesian character" is not to be ignored. As we shall see when this feature is properly understood and incorporated, *a genuine geometric interpretation of quantization can be rigorously developed that agrees*

with the predictions of Schrödinger, Heisenberg, and Feynman, and does so for a wide (dense) set of Hamiltonians.

Coherent States

The concept of coherent states is sufficiently broad by now that there are several definitions. Our definition is really an old one [4], and a very general one at that, so general that it captures essentially all other definitions. We start with a label space $\mathcal{L}$, which may often be identified with the classical phase space M, and a continuous map from points in the label space to (nonzero) vectors in a Hilbert space $\mathbb{H}$ (see below). For concreteness let us choose the label space as the phase space for a single degree of freedom. Then each point in M may be labelled by canonical coordinates (p, q), and we use that very set to identify the coherent state vector itself: $|p, q\rangle$ or $\Phi[p, q]$. If we choose a different set of canonical coordinates, say $(\overline{p}, \overline{q})$ to identify the same point in M, then we associate the new coordinates to the *same* vector $|\overline{p}, \overline{q}\rangle \equiv |p, q\rangle$, or even better $\overline{\Phi}[\overline{p}, \overline{q}] \equiv \Phi[p, q]$. Although it is not necessary to do so, we shall specialize to coherent states that are unit vectors, $\langle p, q|p, q\rangle = 1 = (\Phi[p, q], \Phi[p, q])$, for all points $(p, q) \in M$. We place only two requirements on this map from M into $\mathbb{H}$:

(1) *continuity*, which can be stated as joint continuity in both arguments of the coherent state overlap $\mathcal{K}(p'', q''; p', q') \equiv \langle p'', q''|p', q'\rangle$; and

(2) *resolution of unity*, for which a positive measure μ exists such that

$$\mathbb{1} \equiv \int |p, q\rangle\langle p, q| \, d\mu(p, q) \, ,$$

where $\mathbb{1}$ is the unit operator. This last equation may be understood as

$$\begin{aligned} \langle\phi|\psi\rangle &= \int \langle\phi|p, q\rangle\langle p, q|\psi\rangle \, d\mu(p, q) \, , \\ \langle p'', q''|\psi\rangle \equiv \psi(p'', q'') &= \int \mathcal{K}(p'', q''; p, q) \, \psi(p, q) \, d\mu(p, q) \, , \\ \mathcal{K}(p'', q''; p', q') &= \int \mathcal{K}(p'', q''; p, q) \, \mathcal{K}(p, q; p', q') \, d\mu(p, q) \, . \end{aligned}$$

Each successive relation has been obtained from the previous one by specialization of the vectors involved. The last relation, in conjunction with the fact that $\mathcal{K}(p'', q''; p', q')^* = \mathcal{K}(p', q'; p'', q'')$, implies that $\mathcal{K}$ is the kernel

of a projection operator onto a proper subspace of $L^2(\mathbb{R}^2, d\mu)$ composed of bounded, continuous functions that comprise the Hilbert space of interest. The fact that the representatives are bounded follows from our choice of coherent states that are all unit vectors. Although no group need be involved in our definition of coherent states, it is evident that when a group is present simplifications may occur. This is true for the canonical coherent states to which we now specialize.

With Q and P self adjoint and irreducible and $[Q,P]=i\hbar$, the canonical coherent states defined with help of the unitary Weyl group operators for all $(p,q)\in\mathbb{R}^2\equiv M$ by

$$|p,q\rangle = e^{-iqP/\hbar}\,e^{ipQ/\hbar}\,|\eta\rangle\;, \qquad \langle\eta|\eta\rangle = 1\;;$$

are a standard example for which $d\mu(p,q)=dp\,dq/2\pi\hbar$. The resolution of unity holds in this case for *any* normalized fiducial vector $|\eta\rangle$. However, a useful specialization occurs if we insist that $(\Omega Q+iP)|\eta\rangle=0$, $\Omega>0$, leading to the ground state of an harmonic oscillator. In that case

$$\langle p',q'|p,q\rangle = \exp\{(i/2\hbar)(p'+p)(q'-q)-(1/4\hbar)[\Omega^{-1}(p'-p)^2+\Omega(q'-q)^2]\}\;.$$

We note in passing that the more usual resolutions of unity may be obtained as limits. In particular,

$$\int \lim_{\Omega\to\infty} |p,q\rangle\langle p,q|\,dp\,dq/2\pi\hbar = \int |q\rangle\langle q|\,dq = 1\!\!1\;,$$
$$\int \lim_{\Omega\to 0} |p,q\rangle\langle p,q|\,dp\,dq/2\pi\hbar = \int |p\rangle\langle p|\,dp = 1\!\!1\;,$$

where the formal vectors $|q\rangle$ satisfy $Q|q\rangle = q|q\rangle$and $\langle q'|q\rangle=\delta(q'-q)$, and correspondingly for $|p\rangle$.

The normalized canonical coherent states $|p,q\rangle$ that follow from the condition $(\Omega Q+iP)|\eta\rangle=0$ are in fact analytic functions of the combination $\Omega q+ip$ apart from a common prefactor. When that prefactor is removed from the vectors and put into the integration measure, one is led directly to the Segal-Bargmann representation by holomorphic functions.

Symbols

Generally, and with the notation $\langle(\cdot)\rangle\equiv\langle\eta|(\cdot)|\eta\rangle$, it follows from the commutation relations that $\langle p,q|P|p,q\rangle = p+\langle P\rangle$ and $\langle p,q|Q|p,q\rangle = q+\langle Q\rangle$;

if $\langle p,q|P|p,q\rangle = p$ and $\langle p,q|Q|p,q\rangle = q$ we say that the fiducial vector is physically centered. Observe that the labels of the coherent state vectors are *not eigenvalues* but *expectation values*; thus there is no contradiction in specifying both p and q simultaneously.

For a general operator $\mathcal{H}(P,Q)$ we introduce the *upper* symbol

$$\begin{aligned} H(p,q) &\equiv \langle p,q|\mathcal{H}(P,Q)|p,q\rangle \\ &= \langle \eta|\mathcal{H}(P+p,Q+q)|\eta\rangle \\ &= \mathcal{H}(p,q) + \mathcal{O}(\hbar;p,q)\ , \end{aligned}$$

and, when it exists, the *lower* symbol $h(p,q)$ implicitly defined through the relation

$$\mathcal{H} = \int h(p,q)\,|p,q\rangle\langle p,q|\,dp\,dq/2\pi\hbar\ .$$

We note that for the harmonic oscillator fiducial vector lower symbols exist for a dense set of operators, and generally $H(p,q) - h(p,q) \simeq O(\hbar)$. The association of an operator $\mathcal{H}$ with the function $h(p,q)$ is an example of what goes under the name of Toeplitz quantization today [5].

Differentials

Several differential expressions are already implicitly contained within the coherent states. The first is the canonical one form

$$\theta \equiv i\hbar\langle\ |d|\ \rangle = i\hbar(\Phi, d\Phi) = i\hbar\Sigma_n\phi_n^*\,d\phi_n$$

written in coordinate-free notation, or alternatively,

$$\theta = i\hbar\langle p,q|d|p,q\rangle = p\,dq + \langle P\rangle\,dq - \langle Q\rangle\,dp = p\,dq$$

using canonical coordinates, and where we have ended with a physically centered fiducial vector. In coordinate-free notation it follows that

$$\omega \equiv d\theta = i\hbar\Sigma_n d\phi_n^* \wedge d\phi_n\ ,$$

and in canonical coordinates that

$$\omega = dp\wedge dq = d\overline{p}\wedge d\overline{q}\ ,$$

along with $d\omega = 0$ which follows directly. A useful Riemannian metric is given first in coordinate-free notation by

$$\begin{aligned} d\sigma^2 &\equiv 2\hbar^2[\|d|\,\rangle\|^2 - |\langle\,|d|\,\rangle|^2] \\ &= 2\hbar^2\Sigma_{n,m}\, d\phi_n^*(\delta_{nm} - \phi_n\phi_m^*)d\phi_m \ ; , \end{aligned}$$

and second in canonical coordinates by

$$\begin{aligned} d\sigma^2(p,q) &= \hbar(dp^2 + dq^2)\ , \qquad (\Omega = 1)\ , \\ d\sigma^2(\overline{p},\overline{q}) &= \hbar[A(\overline{p},\overline{q})d\overline{p}^2 + B(\overline{p},\overline{q})d\overline{p}\,d\overline{q} + C(\overline{p},\overline{q})d\overline{q}^2]\ . \end{aligned}$$

In the next to the last line the line element is expressed in the Cartesian form it takes for a Gaussian fiducial vector, while in the last line is the expression of the flat metric in general canonical coordinates.

Canonical and unitary transformations

In classical mechanics canonical transformations may either be viewed as passive or active. Passive transformations leave the point in phase space fixed but change the coordinates by which it is described; active transformations describe a flow of points in phase space against a fixed coordinate system. The best known example of an active transformation is the continuous unfolding in time of a dynamical evolution. In quantum mechanics unitary transformations are presumed to play the role that canonical transformations play in the classical theory [6]. If $p \rightarrow P$ and $q \rightarrow Q$, then it follows that $\overline{p} = (p+q)/\sqrt{2} \rightarrow (P+Q)/\sqrt{2} \equiv \overline{P}$ and $\overline{q} = (q-p)/\sqrt{2} \rightarrow (Q-P)/\sqrt{2} \equiv \overline{Q}$, and moreover there exists a unitary operator U such that $\overline{P} = U^\dagger PU$ and $\overline{Q} = U^\dagger QU$. Consider instead the classical canonical transformation $\tilde{p} \equiv (p^2+q^2)/2 \rightarrow \tilde{P}$ and $\tilde{q} \equiv \tan^{-1}(q/p) \rightarrow \tilde{Q}$. As basically a transformation to polar coordinates this canonical transformation is well defined except at the single point $p = q = 0$. However, the associated quantum operators in this case cannot be connected by a unitary transformation to the original operators P and Q (because $\tilde{P} \geq 0$ and the spectrum of an operator is preserved under a unitary transformation). Thus some passive canonical transformations have images in unitary transformations while others definitely do not.

Using coherent states it is possible to completely disconnect canonical transformations and unitary transformations. Consider the transformations

of the upper and lower symbols in the following example:

$$\begin{aligned}
\tfrac{1}{2}(p^2+q^2) &= \langle p,q|\,\tfrac{1}{2}(P^2+Q^2-\hbar)|p,q\rangle \\
&= \langle \tilde{p},\tilde{q}|\,\tfrac{1}{2}(P^2+Q^2-\hbar)|\tilde{p},\tilde{q}\rangle = \tilde{p}\ , \\
\tfrac{1}{2}(P^2+Q^2+\hbar) &= \int \tfrac{1}{2}(p^2+q^2)\,|p,q\rangle\langle p,q|\,dp\ ,dq/2\pi\hbar \\
&= \int \tilde{p}\,|\tilde{p},\tilde{q}\rangle\langle\tilde{p},\tilde{q}|\,d\tilde{p}\,d\tilde{q}/2\pi\hbar\ .
\end{aligned}$$

Observe in this example how the operators and coherent state vectors have remained completely fixed as the coordinates have passed from (p,q) to $(\tilde{p},\tilde{q})$. Of course, one may also introduce separate and arbitrary unitary transformations of the operators and vectors, e.g. $P \to VPV^\dagger$, and $|p,q\rangle \to V|p,q\rangle$, etc., which have the property of preserving inner products.

Shadow Metric and Cartesian Coordinates

The form of the metric $d\sigma^2$ was given earlier for a special (harmonic oscillator) fiducial vector. If instead we consider a general fiducial vector $|\eta\rangle$, then it follows that

$$d\sigma^2 = \langle(\Delta Q)^2\rangle\,dp^2 + \langle \Delta P\Delta Q + \Delta Q\Delta P\rangle\,dp\,dq + \langle(\Delta P)^2\rangle\,dq^2\ ,$$

which shows itself to be always *flat*; thus this is a property of the *Weyl group* and not of the fiducial vector. Here $\Delta P \equiv P - \langle P\rangle$, etc. Unlike the symplectic form or the Hamiltonian, for example, the metric is typically $O(\hbar)$ and thus it is essentially nonclassical.

Indeed, any quantization scheme in which the Weyl operators and Hilbert space vectors appear leads to the metric $d\sigma^2$, whether it is intentional or not. Such schemes may not *use* the metric, but it is nevertheless there.

We assert that physics resides in Cartesian coordinates, and more particularly in the coordinate form of the metric. Suppose $d\sigma^2 = \hbar(dp^2+dq^2)$, then it follows that $H(p,q) = \frac{1}{2}(p^2+q^2)$ implies, as before, that $\mathcal{H} = \frac{1}{2}(P^2+Q^2-\hbar)$. On the other hand, if instead $d\sigma^2 = \hbar[(2\tilde{p})^{-1}d\tilde{p}^2 + (2\tilde{p})d\tilde{q}^2]$, then $\tilde{H}(\tilde{p},\tilde{q}) = \tilde{p}$ implies that $\mathcal{H} = \frac{1}{2}(P^2+Q^2-\hbar)$. In other words, *the physical meaning of the coordinatized mathematical expression for some classical quantity is coded into the coordinate form of the metric!* This remark is already true at the

classical level, namely one needs a "shadow" flat metric on the classical phase space, or at least on a copy of it, so that one can ascribe physical meaning to the coordinatized mathematical expressions for one or another classical quantity. If the flat shadow metric is expressed in Cartesian coordinates, then one may interpret an expression such as $\frac{1}{2}(p^2+q^2)+q^4$ as truly representing a physical, quartic anharmonic oscillator; if the flat shadow metric is *not* expressed in Cartesian coordinates, then no such physical interpretation of such a mathematical expression is justified.

Although we have originally introduced the phase-space metric in the quantum theory via its construction in terms of coherent states, we now see that we can alternatively view the phase-space metric (modulo a coefficient $\hbar$) as an auxiliary classical expression that provides physical meaning for coordinatized expressions of the classical theory.

Quantization and Continuous-time Regularization

It should be self evident that quantization relates to physical systems inasmuch as the quantization of a particular Hamiltonian is designed to generate the spectrum appropriate to that physical system. Consider again the formal phase-space path integral given by

$$\mathcal{M}\int e^{(i/\hbar)\int_0^T [p\dot{q}+\dot{G}(p,q)-h(p,q)]\,dt}\,\mathcal{D}p\,\mathcal{D}q\ .$$

We have already stressed that this expression is not mathematically defined, and now we emphasize that in fact it has no physics as well because there is no way of telling to which physical system the coordinatized expression for the Hamiltonian corresponds. In short, the formal phase-space path integral expression has neither mathematical nor physical meaning as it stands!

We will remedy this situation in a moment, but there is one "toy" analog worth introducing initially. Consider the conditionally convergent integral that is given a definition through the introduction of a regularization and its removal as in the expression

$$\int_{-\infty}^{\infty} e^{iy^2/2}\,dy \equiv \lim_{\nu\to\infty}\int_{-\infty}^{\infty} e^{iy^2/2-y^2/2\nu}\,dy=\sqrt{2\pi i}\ .$$

Other regularizations may lead to the same answer, or in fact they may lead to different results; the physical situation should be invoked to choose the relevant one.

Now let us introduce a continuous-time convergence factor into the formal phase-space path integral in the form

$$\lim_{\nu\to\infty} \mathcal{M}_\nu \int e^{(i/\hbar)\int_0^T [p\dot{q}+\dot{G}(p,q)-h(p,q)]\,dt}\, e^{-(1/2\nu)\int_0^T (\dot{p}^2+\dot{q}^2)\,dt}\,\mathcal{D}p\,\mathcal{D}q$$
$$= \lim_{\nu\to\infty} 2\pi\hbar\, e^{\hbar\nu T/2} \int e^{(i/\hbar)\int_0^T [p\,dq+dG(p,q)-h(p,q)\,dt]}\, d\mu_W^\nu(p,q)\ .$$

In the first line we have formally stated the form of the regularization, while in the second line appears the proper mathematical statement it assumes after some minor rearrangement. The measure μ_W^ν is a two-dimensional Wiener measure expressed in Cartesian coordinates on the plane as signified by the metric dp^2+dq^2 that appears in the first line in the regularization factor. *Here enters the very shadow metric itself, used to give physical meaning to the coordinatized form of the Hamiltonian, and which now additionally underpins a rigorous regularization for the path integral!* As Brownian motion paths, with diffusion constant ν, almost all paths are continuous but nowhere differentiable. Thus the initial term $\int p\,dq$ needs to be defined as a *stochastic integral* and we choose to do so in the Stratonovich form, namely as $\lim \Sigma\frac{1}{2}(p_{l+1}+p_l)(q_{l+1}-q_l)$, where $q_l \equiv q(l\epsilon)$, etc., and the limit refers to $\epsilon\to 0$ [7]. This prescription is generally different from that of Itô, namely $\lim \Sigma p_l(q_{l+1}-q_l)$, due to the unbounded variation of the Wiener paths involved. Observe, in the second line above, for each $0<\nu<\infty$, that *no mathematical ambiguities remain*, i.e., the expression is completely well defined. As we note below not only does the limit exist but it also provides the correct solution to the Schrödinger equation for a dense set of Hamiltonian operators.

The continuous-time regularization involved, or its Wiener measure counterpart, involves pinning the paths $p(t), q(t)$ at $t=T$ and at $t=0$ so that $(p'',q'')=(p(T),q(T))$ and $(p',q')=(p(0),q(0))$. This leads to an expression of the form $K(p'',q'',T;p',q',0)$, which may be shown to be

$$K(p'',q'',T;p',q',0) \equiv \langle p'',q''|\, e^{-i\mathcal{H}T/\hbar}|p',q'\rangle\ ,$$
$$|p,q\rangle \equiv e^{-iG(p,q)/\hbar}\, e^{-iqP/\hbar}\, e^{ipQ/\hbar}\,|\eta\rangle\ , \quad (Q+iP)|\eta\rangle = 0\ ,$$
$$\mathcal{H} \equiv \int h(p,q)|p,q\rangle\langle p,q|\,dp\,dq/2\pi\hbar\ .$$

In brief, the regularization chosen *automatically* leads to a coherent state representation, and, in addition, it *selects* the Hamiltonian operator determined by the lower symbol. There are three technical requirements for this representation to hold [8]:

$$
\begin{array}{lll}
(1) & \int h^2(p,q)\, e^{-A(p^2+q^2)}\, dp\, dq < \infty \;, & \text{for all } \; A > 0 \;, \\
(2) & \int h^4(p,q)\, e^{-B(p^2+q^2)}\, dp\, dq < \infty \;, & \text{for some } \; B < \tfrac{1}{2} \;, \\
(3) & \mathcal{H} \;\; \text{is e.s.a. on } \; D = \{\Sigma_0^N a_n |n\rangle :\; a_n \in \mathbf{C},\; N < \infty\} \;, &
\end{array}
$$

where the orthonormal states $|n\rangle \equiv (1/\sqrt{n!})[(Q - iP)/\sqrt{2\hbar}]^n |\eta\rangle$, $n \geq 0$. Thus this representation includes (but is not limited to) *all Hamiltonians that are Hermitian, semibounded polynomials of the basic operators P and Q*. We note that G generally serves as an unimportant gauge; however, if the topology of M is not simply connected then G contains the Aharanov-Bohm phase [9]. Observe that the propagator formula also has an *analog physical system*, namely a two-dimensional particle moving on a flat plane in the presence of a constant magnetic field perpendicular to the plane. The limit in which the mass of the particle goes to zero projects the system onto the first Landau level.

The point of using the Stratonovich prescription for stochastic integrals is that the ordinary rules of calculus still apply [7]. Thus the rule for a canonical transformation given earlier, namely $\overline{p}\, d\overline{q} = p\, dq + dF(\overline{q}, q)$, still applies to Brownian motion paths. Consequently, just as in the classical case a function $\overline{G}(\overline{p}, \overline{q})$ exists so that after such a canonical coordinate transformation

$$
\begin{aligned}
\overline{K}(\overline{p}'', \overline{q}'', T; \overline{p}', \overline{q}', 0) &= \langle \overline{p}'', \overline{q}''|\, e^{-i\mathcal{H}T/\hbar}\, |\overline{p}', \overline{q}'\rangle \\
&= \lim_{\nu\to\infty} 2\pi\hbar\, e^{\hbar\nu T/2} \int e^{(i/\hbar)\int_0^T [\overline{p}d\overline{q} + d\overline{G}(\overline{p},\overline{q}) - \overline{h}(\overline{p},\overline{q})dt]}\, d\overline{\mu}_W^{\nu}(\overline{p}, \overline{q}) \;.
\end{aligned}
$$

Here $\overline{\mu}_W^{\nu}$ denotes two-dimensional Wiener measure on the flat plane expressed in general canonical coordinates.

Coordinate-free formulation

The covariant transformation of the propagator indicated above implies that a coordinate-free representation exists. We first introduce Brownian motion as a map $\rho(t; 0) : M \times M \to \mathbb{R}^+$, $t > 0$, with $\lim_{t\to 0} \rho = \delta$, $\partial\rho/\partial t =$

$(\nu/2)\Delta\rho$, and finally $\rho(t;0) = \int d\mu_W^\nu$ which defines a coordinate-free Wiener measure. Next we let $\phi : M \to \mathbf{C}$, $\mathcal{K} : M \times M \to \mathbf{C}$, $\phi \in (\mathcal{K})L^2(M,\omega)$, and $\phi = \mathcal{K}\phi$ (N.B. $\mathcal{K}$ is an analog of "polarization"). Quantum dynamics comes from $i\hbar\partial\phi/\partial t = \mathcal{H}\phi$, where $\mathcal{H} = \mathcal{K}h\mathcal{K}$ (N.B. this relation has the effect of "preserving polarization"); also we introduce $K(T;0) : M \times M \to \mathbf{C}$, so that $\phi(T) = K(T;0)\phi(0)$. The construction of the reproducing kernel $\mathcal{K}$ and the propagator K reads

$$\begin{aligned}
\mathcal{K} \;&\equiv\; \lim_{\nu\to\infty} 2\pi\hbar\, e^{\hbar\nu T/2} \int e^{(i/\hbar)\int(\theta + dG)}\, d\mu_W^\nu = \lim_{T\to 0} K(T;0)\ , \\
K(T;0) \;&\equiv\; \lim_{\nu\to\infty} 2\pi\hbar\, e^{\hbar\nu T/2} \int e^{(i/\hbar)\int(\theta + dG - h\,dt)}\, d\mu_W^\nu\ .
\end{aligned}$$

Observe again how a flat metric has been used for the Brownian motion; in our view it is this flat (phase) space that underlies Dirac's remark related to canonical quantization quoted above.

Alternative continuous-time regularizations

Our introduction of Brownian motion on a *flat* two-dimensional phase space has led to canonical quantization, namely one involving the Heisenberg operators P and Q. If instead we choose to regularize on a phase space taken as a two-dimensional *spherical* surface of radius R, where $R^2 \equiv s = \hbar/2,\ \hbar,\ 3\hbar/2, \ldots$, then such a Brownian motion regularization leads to a quantization in which the kinematical operators are the spin operators $S_1,\ S_2$, and S_3 such that $\Sigma S_j^2 = s(s+1)\hbar^2$, i.e., the generators of the SU(2) group [8]. In like manner, if we introduce a Brownian motion regularization on a two-dimensional *pseudo-sphere* of constant negative curvature, then the kinematical operators that emerge are the generators of the affine ("$ax+b$") group, a subgroup of SU(1,1)[10]. The three examples given here exhaust the simply connected spaces of constant curvature in two dimensions; they also have the property that the metric assumed for the Brownian motion regularization coincides with the metric that follows from the so-derived coherent states.

Summarizing, *the geometry of the regularization that supports the Brownian motion actually determines the nature of the kinematical operators in the quantization!*

Regularization on a General 2-D Surface

Finally, we note that the present kind of quantization can be extended to a general two-dimensional surface without symmetry and with an arbitrary number of handles. We only quote the result for the propagator. Let $\xi^j,\ j=1,2$, denote the two coordinates, $g_{jk}(\xi)$ the metric, $a_j(\xi)$ a two-vector, $f_{jk}(\xi)=\partial_j a_k(\xi)-\partial_k a_j(\xi)$ its curl, and $h(\xi)$ the classical Hamiltonian. Then the propagator is defined by [11]

$$\begin{aligned}
\langle\xi'',T|\xi',0\rangle &= \langle\xi''|\,e^{-i\mathcal{H}T/\hbar}|\xi'\rangle \\
&= \lim_{\nu\to\infty}\mathcal{M}_\nu\int\exp\{(i/\hbar)\int[a_j(\xi)\dot\xi^j-h(\xi)]\,dt\} \\
&\quad\times\exp\{-(1/2\nu)\int g_{jk}(\xi)\dot\xi^j\dot\xi^k\,dt\} \\
&\quad\times\exp\{(\hbar\nu/4)\int\sqrt{g(\xi)}\epsilon^{jk}f_{jk}(\xi)\,dt\}\;\;Pi_t\sqrt{g(\xi)}\,d\xi^1\,d\xi^2\ .
\end{aligned}$$

Observe, in this general setting, that the phase-space metric tensor $g_{jk}(\xi)$ is one of the necessary *inputs* to the process of quantization under discussion. From this viewpoint the phase-space metric induced by the coherent states is regarded as a derived quantity, and in the general situation the two metrics may well differ. For a compact manifold M it is necessary that $\int f_{jk}(\xi)d\xi^j\wedge d\xi^k=4\pi\hbar n$, $n=1,2,3,\ldots$. In this case the Hilbert space dimension $D=n+1-\overline{g}$, where $\overline{g}$ is the number of handles in the space (genus). Here the states $|\xi\rangle$ are coherent states that satisfy

$$\begin{aligned}
\mathbb{1} &= \int|\xi\rangle\langle\xi|\,\sqrt{g(\xi)}\,d\xi^1\,d\xi^2\ , \\
\mathcal{H} &= \int h(\xi)\,|\xi\rangle\langle\xi|\,\sqrt{g(\xi)}\,d\xi^1\,d\xi^2\ .
\end{aligned}$$

Observe that although the states $|\xi\rangle$ are coherent states in the sense of this article, there is generally *no transitive group* with which they may be defined. The propagator expression above is manifestly covariant under arbitrary coordinate transformations, and a gauge transformation of the vector a introduces a gauge-like contribution that does not appear in the field f. Finally—and contrary to general wisdom—we note that the weighting in the case of a general geometry is *nonuniform* in the sense that the symplectic form $\omega=f_{jk}\,d\xi^j\wedge d\xi^k/2$ is generally *not* proportional to the volume element

$\sqrt{g(\xi)}\, d\xi^1\, d\xi^2$ needed in the resolution of unity and hence in the path integral construction.

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